Tune Shifts Caused by Horizontal Closed Orbit Deviations in Sextupoles

I. Introduction

One of the uncomfortable features of the Chasman-Green lattice is that the chromaticity-correcting sextupoles are all very strong compared with those in the FODO-type lattice. Because of their strengths, when their arrangement creates certain harmonic components, the dynamic aperture is severely reduced and one is forced to add more sextupoles to eliminate harmful harmonic components. In the 7-GeV ring, four sextupoles are planned in each cell for this purpose in addition to three per cell for controlling chromaticities.

1. harmonic sextupoles $(B''\ell/B\rho)$ in $(\text{meters})^{-2}$

      S_1 (two/cell)   1.902

      S_2 (two/cell)   -3.696

2. chromaticity sextupoles

      S_D (two/cell)   -4.266

      S_F (one/cell)   3.960

It is well-known that vertical closed orbit deviations in sextupoles effectively create skew quadrupole field which enhances the linear horizontal-vertical coupling of betatron motion. Horizontal orbit deviations, on the other hand, create a shift in tunes in both transverse directions. Since the shift will be coherent, i.e., common to all particles in a beam, it should not be serious as long as tunes can be readjusted. During the commissioning, sextupoles are expected to be all off until a good closed orbit is

established. The problem is then reduced to how well one should align the position monitor relative to the field center of adjacent sextupole magnet.

During the course of design studies, S. Kramer has made many computer runs to investigate tune shifts resulting from horizontal orbit deviations in sextupoles.[1] Recently, W. Chou explained how to estimate the expected rms value of tune shifts when closed orbit deviations are caused exclusively by horizontal misalignment of quadrupoles.[2] He points out that, under such a condition, the deviations in sextupoles are highly correlated to each other so that the statistical averaging must be over the random quadrupole misalignments and not over the (not-at-all random) orbit deviations in sextupoles. He has demonstrated that the analytical prediction of the rms tune shift agrees fairly well with numerical results obtained by S. Kramer and by Y. Jin.

A feature in these numerical results that has been noticed by Kramer remains unexplained in the analysis by Chou who states that " ..... these programs give a finite average tune shift $\overline{\Delta\nu}$ in addition to an rms $\langle\Delta\nu\rangle$ ..... ". To be sure, only twenty-four random samples are cited in his report and the values of $\overline{\Delta\nu}$ are rather small, ranging from -0.007 to -0.058 horizontally and from 0.009 to 0.017 vertically. According to Kramer, however, this small but finite average tune shift persists even with many more random samples. An interesting observation by him is that the average is definitely related to the dependence of tunes on the betatron oscillation amplitudes (or, equivalently, the transverse emittances). Kramer says some people have even cast doubts on the reliability of computer programs because of the seemingly remote (if any) connection.

This note is an "attempt" to explain the connection at least qualitatively. It is no more than an attempt since the explanation is not yet

quantitative and it may even be somewhat inconsistent. One by-product of this work is the numerical evaluation of amplitude-dependence of tunes in the second order of sextupole strength. There are three coefficients and they are given in the latest CDR.[3] The estimate given in this note agrees very well with the CDR result for one (main) coefficient but disagrees for the other two. This is the case either with or without harmonic sextupoles S_1 and S_2. Explicit analytical expressions used for the estimate are given, both in closed forms and as infinite series in terms of harmonic components. It is hoped that this note will be of some help to those who may be interested in pursuing the questions raised here.

II. Tune and Emittance: Analytical Results

By now, the second-order sextupole effects are common knowledge but the analytical expressions are not always presented in convenient forms.

Although no standard forms that are accepted by everyone as such exist, there are two ways to express the amplitude dependence of tunes which arises in the second order of sextupole strength, one in closed forms and the other as infinite series. For most numerical purposes, the closed forms are naturally more convenient but the infinite series in terms of harmonic components can show more clearly "what is going on". That these two entirely different expressions are mathematically equivalent has been shown explicitly by K. Y. Ng.[4] The closed forms cited in his report contain the so-called distortion functions that have been defined by Tom Collins in connection with the distortion of beam shape in phase space.[5] Forms given below intentionally avoid the use of distortion functions for those who are not familiar with the special symbols.

$$\Delta \nu_x = X\varepsilon_x + Z\varepsilon_y; \qquad \Delta \nu_y = Y\varepsilon_y + Z\varepsilon_x$$

where ε_x, ε_y, = horizontal and vertical beam emittances.

$$X = -\{3/((128\,\pi\,\sin(\pi\nu_x))\}\,\sum s_k\,s_j\,\cos(|\psi_x^{(k)} - \psi_x^{(j)}| - \pi\nu_x)$$
$$-\{\,1/(128\,\pi\,\sin(3\pi\nu_x))\}\,\sum s_k\,s_j\,\cos(3|\psi_x^{(k)} - \psi_x^{(j)}| - 3\pi\nu_x); \qquad (2)$$

$$Z = \{1/(32\,\pi\,\sin(\pi\nu_x))\}\,\sum s_k\,\bar{s}_j\,\cos(|\psi_x^{(k)} - \psi_x^{(j)}| - \pi\nu_x)$$
$$-\{1/64\,\pi\,\sin(\pi\nu_\pm))\}\,\sum \bar{s}_k\,\bar{s}_j\,\cos(|\psi_\pm^{(k)} - \psi_\pm^{(j)}| - \pi\nu_\pm); \qquad (3)$$

$$Y = -\{1/(32\,\pi\,\sin(\pi\nu_x))\}\,\sum \bar{s}_k\,\bar{s}_j\,\cos(|\psi_x^{(k)} - \psi_x^{(j)}| - \pi\nu_x)$$
$$\mp\{1/(128\,\pi\,\sin(\pi\nu_\pm))\}\,\sum \bar{s}_k\,\bar{s}_j\,\cos(|\psi_\pm^{(k)} - \psi_\pm^{(j)}| - \pi\nu_\pm). \qquad (4)$$

ψ_x, ψ_y = betatron phase angles (0 to $2\pi\,\nu_{x,y}$),

$$\psi_\pm = 2\psi_y \pm \psi_x$$
$$\nu_\pm = 2\nu_y \pm \nu_x$$
$$s_k = (\beta_x^{3/2})_k\,(B''\ell/B\rho)_k; \qquad \bar{s}_k = (\sqrt{\beta_x}\,\beta_y)_k\,(B''\ell/B\rho)_k$$

In all these formulas, double summations should be

$$\sum_{k=1}^{N}\,\sum_{j=1}^{N}; \qquad N = \text{total number of sextupoles in the ring.}$$

For the 7-GeV APS ring, the summation over (j) should be for all (7x40=280) sextupoles but the summation over (k) can be replaced by 40 times summation over seven sextupoles in one cell. In the above expressions for X, Y and Z, the terms related to the orbit distortions are clearly the first of two lines with the familiar expression

$$\cos(|\psi_x^{(k)} - \psi_x^{(j)}| - \pi\nu_x)/\sin(\pi\nu_x)$$

Expressions for X, Y and Z in terms of harmonic components are given in Ref. 6, and only the one for X is given below.

$$X = 54 \sum_m \frac{A^2_{1m}}{m-\nu_x} + 18 \sum_m \frac{A^2_{3m}}{m-3\nu_x} \qquad (5)$$

where the summation is for $m = -\infty$ to $+\infty$. Harmonic components A_{1m} and A_{3m} are given by

$$A_{1m} = (1/48\pi) \; |\sum_k s_k \exp[i(\psi_x^{(k)} - \nu_x\theta_k + m\theta_k)]|, \qquad (6)$$

$$A_{3m} = (1/48\pi) \; |\sum_k s_k \exp[i(3\psi_x^{(k)} - 3\nu_x\theta_k + m\theta_k)]| \qquad (7)$$

Summations over (k) are for all sextupoles in the ring but they are equal to 40x(summations over seven sextupoles in a cell). Because of the mirror symmetry of lattice, both A_{1m} and A_{3m} are real. Moreover, they are all zero except for those with $m = 0$, ±40, ±80, etc. The connection to the orbit deviations in sextupoles comes from the fact that, for the 7-GeV ring, the term with $A_{1,40}$ is the most important in (5); of all possible values of m, 40 is the closest to $\nu_x = 35.216$.

Numerical values of coefficients X, Y and Z can be found easily from Eq. (2) through Eq. (4) once the lattice and sextupoles are specified. In

order to compare with results given in the CDR, tune shifts are expressed as a function of N_x and N_y, keeping in mind that

$$\varepsilon_x = (8 \times 10^{-9} \text{ m}) \times N_x^2 \qquad \text{and} \qquad \varepsilon_y = (4 \times 10^{-9} \text{ m}) \times N_y^2 \qquad (8)$$

1. Chromaticity-correcting sextupoles only (S_D and S_F)

$$\Delta\nu_x = 2.85 \times 10^{-4} \ N_x^2 \qquad + 0.29 \times 10^{-4} \ N_y^2$$

$$\Delta\nu_y = 0.57 \times 10^{-4} \ N_x^2 \qquad + 0.55 \times 10^{-4} \ N_y^2$$

From CDR, p. II.1-13,

$$\Delta\nu_x = 2.87 \times 10^{-4} \ N_x^2 \qquad + 0.60 \times 10^{-4} \ N_y^2$$

$$\Delta\nu_y = 1.20 \times 10^{-4} \ N_x^2 \qquad + 0.39 \times 10^{-4} \ N_y^2$$

2. With harmonic sextupoles (S_D, S_F, S_1, S_2)

$$\Delta\nu_x = 6.00 \times 1^{-7} \ N_x^2 \qquad - 0.74 \times 10^{-4} \ N_y^2$$

$$\Delta\nu_y = -1.48 \times 1^{-4} \ N_x^2 \qquad + 0.50 \times 10^{-4} \ N_y^2$$

From CDR, p. II.1-17,

$$\Delta\nu_x = 0.00 \quad N_x^2 \qquad + 0.72 \times 10^{-5} \ N_y^2$$

$$\Delta\nu_y = 1.44 \times 10^{-5} \ N_x^2 \qquad + 0.98 \times 10^{-5} \ N_y^2$$

With or without harmonic sextupoles, the agreement on $d(\Delta\nu_x)/d(N_x^2)$ is very good but the disagreements for other coefficients are not trivial.

Independent checks of Eq. (2) through Eq. (4) together with their numerical re-evaluations are desirable in order to remove any uncertainties.

It should be noted here that the amplitude dependence of tunes discussed so far is based on perturbation in its lowest order (which is second-order in sextupole strength). If tunes are found as a function of amplitudes through numerical beam tracking and Fourier transform of its results, the dependence may not agree with the lowest-order analytical estimates presented above, especially for large amplitudes approaching the stable boundary of some resonance.

III. Possible Source of Average Tune Shift

If linear coupling of transverse betatron motions is introduced through the effective skew quadrupole component, one must deal with two eigentunes instead of ν_x and ν_y and this complicates the analysis. It is therefore assumed here that closed orbit deviations in sextupoles are in the horizontal direction only. This is not unphysical; one can in principle (if not in practice) establish a purely horizontal motion as long as the field is "normal" (i.e., no skew component). Convention for suffix used throughout this section is:

quadrupoles = i, j, k (10x40 in the ring)

sextupoles = m, n (3x40 or 7x40 in the ring)

Since horizontal direction alone is considered, suffix to distinguish horizontal from vertical will be omitted.

Orbit deviation $(\Delta x)_m$ in the m-th sextupole is caused predominantly by δ-function kicks at misaligned quadrupoles but there may be similar kicks

by sextupoles contributing to $(\Delta x)_m$. One writes, for the normalized deviation $\xi_m = (\Delta x)_m/\sqrt{\beta_m}$,

$$\xi_m = \sum_i A_{mi}\Delta_i + \sum_n B_{mn}\xi_n^2 \tag{9}$$

with

$$A_{mi} = (1/2\sin\langle\pi\nu\rangle) \ g_i \ \cos(|\psi_m - \psi_n| - \pi\nu) \tag{10}$$

$$B_{mn} = (1/2\sin\langle\pi\nu\rangle)(-s_n/2)\cos(|\psi_m - \psi_n| - \pi\nu) \tag{11}$$

$$g_i = \sqrt{\beta_i}(B'\ell/B\rho) \quad , \quad s_n = (\beta_n)^{3/2} \ (B''\ell/B\rho)_n \tag{12}$$

In Eq. (9), for small orbit deviations, one may use the lowest-order expression

$$\xi_n^2 = \sum_j A_{nj} \ \Delta_j \sum_k A_{nk} \ \Delta_k \tag{13}$$

The tune shift arising from the m-th sextupole is

$$(\Delta\nu)_m = b_m\xi_m \tag{14}$$

with $b_m = (s_m/4\pi)$. In the lowest order, contributions from many sextupoles are simply added (M = total number of sextupole magnets in the ring),

$$\Delta\nu = \sum_m b_m \xi_{m_i} = \sum_i c_i \ \Delta_i + \sum_n d_n \sum_j A_{nj} \ \Delta_j \sum_k A_{nk} \ \Delta_k \tag{15}$$

where

$$c_i = \sum_m b_m A_{mi} \quad \text{and} \quad d_n = \sum_n b_m B_{mn} \tag{16}$$

Remembering that $(\Delta_i, \Delta_j, \Delta_k)$ are all random misalignments of quadrupoles, one can perform average over many sample cases. The average tune shift is

$$\overline{\Delta\nu} = [\sum_n d_n \sum_j A_{nj}^2]\Delta^2 \tag{17}$$

where Δ is the rms value of quadrupole misalignments Δ_i. From Eq. (10),

$$D_n \equiv \sum_j A_{nj}^2 = (1/4\sin^2\langle\pi\nu\rangle) \sum_j g_j^2 \cos^2(|\psi_n - \psi_j| - \pi\nu) \tag{18}$$

In evaluating this quantity for a given sextupole (n), the summation over (j) must be for all 400 quadrupoles in the ring. One then finds that D_n is practically independent of (n),

$$287 \text{ m}^{-1} < D_n < 294 \text{ m}^{-1}$$

It is thus allowed to use $\bar{D}$ the average value of D_n, in Eq. (17),

$$\Delta\nu \approx \Delta^2 \bar{D} \sum_n d_n$$

$$= -\{\Delta^2\bar{D}/(16\pi\sin\langle\pi\nu\rangle)\} \sum_{m,n} s_m s_n \cos(|\psi_m - \psi_n| - \pi\nu) \tag{19}$$

which is, aside from trivial factors, precisely what one found as a part of $d(\Delta\nu_x)/d(N_x^2)$, the first line of Eq. (2).

If Eq. (19) is taken as something "quantitative", the average tune shift is +0.01 when all sextupoles are used with $\Delta = 1\times10^{-4}$ m. Compared with

the numerical results cited in Ref. 2, the magnitude is not inconsistent but the sign is!

There are at least two shortcomings in the treatment here if its results are to be considered "quantitative" so that they can be compared with results from various computer programs. The first is the ever-present question of coupled vs uncoupled motions and their tunes. In handling results from any computer program, one must be certain what quantities are calculated as "tune". Of course, the uncertainty can be easily eliminated by removing vertical orbit deviations in sextupoles. The second is more difficult and more serious in nature. In estimating the average tune shift $\overline{\Delta\nu}$, contributions from many sextupoles are simply added in Eq. (15). This is not really consistent. Since $\overline{\Delta\nu}$ given in Eq. (17) is proportional to Δ^2, one must include in Eq. (15) contributions coming from many pairs of sextupoles having the form $\propto \xi_m \xi_n$ which is proportional to Δ^2 also. Unless this defect is taken care of, the argument presented in this note cannot be taken as the right explanation of the origin of average tune shift.

References

1. S. Kramer, private communication (1987).

2. W. Chou, Argonne National Laboratory, unpublished information (1987).

3. "7 GeV Advanced Photon Source - Conceptual Design Report", Argonne National Laboratory Report ANL-87-15, (April 1987).

4. K. Y. Ng, Proceedings of the 1984 Summer Study, Snowmass, Colorado (1984); Fermilab TM-1277 and TM-1281.

5. T. Collins, Proceedings of the 1984 Summer Study, Snowmass, Colorado, (1984).

6. S. Ohnuma, Proceedings of the Conference on the Interactions between Particle and Nuclear Physics, Steamboat Springs, Colorado (1984).